An Application of L'Hopital's Rule
Notional Growth Rate is Zero

Gary Schurman MBE, CFA

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L'Hopital's rule can be used when evaluating limits that result in indeterminate forms of (1) zero divided by zero or (2) infinity divided by infinity. In the white paper **Non-Interest-Bearing Liabilities - Growth Rate is Constant** [1], the equation for cumulative investment income recognized over the time interval $[m, n]$ was...

$$I_{m,n} = \frac{\alpha\,(1-\tau)}{\omega}\,\eta\,N_0 \left[\mathrm{Exp}\left\{\omega\,n\right\} - \mathrm{Exp}\left\{\omega\,m\right\}\right] \tag{1}$$

The definition of the parameters in the equation above are...

Symbol	Description
$I_{m,n}$	Cumulative after-tax investment income over time interval [m,n]
N_0	Notional value at time zero
ω	Continuous-time notional value growth rate
α	Continuous-time investment yield
η	Ratio of investment value to notional value
τ	Income tax rate

Note that when the notional value growth rate in Equation (1) above is zero that equation is in the indeterminate form of zero divided by zero. To solve that equation when $\omega = 0$ we will make the following definitions...

$$f(w) = \alpha\,(1-\tau)\,\eta\,N_0 \left[\mathrm{Exp}\left\{\omega\,n\right\} - \mathrm{Exp}\left\{\omega\,m\right\}\right] \text{ ...and... } g(w) = \omega \tag{2}$$

Using the function definitions in Equation (2) above, we can rewrite Equation (1) above as...

$$I_{m,n} = \frac{f(\omega)}{g(\omega)} \text{ ...such that... } \frac{f(0)}{g(0)} = \frac{0}{0} \text{ ...when... } \omega = 0 \tag{3}$$

The derivatives of the functions $f(\omega)$ and $g(\omega)$ in Equation (2) above with respect to the notional value growth rate ω are...

$$f'(\omega) = \frac{\delta\,f(w)}{\delta\omega} = \alpha\,(1-\tau)\,\eta\,N_0 \left[n\,\mathrm{Exp}\left\{\omega\,n\right\} - m\,\mathrm{Exp}\left\{\omega\,m\right\}\right] \text{ ...and... } g'(\omega) = \frac{g(w)}{\delta\omega} = 1 \tag{4}$$

The differentiation of the numerator and denominator converts it to a limit that can be directly evaluated by continuity. Using the equations above, Equation (1) above can be rewritten as...

$$I_{m,n} = \lim_{\omega\to 0} \frac{f(\omega)}{g(\omega)} = \lim_{\omega\to 0} \frac{f'(\omega)}{g'(\omega)} \text{ ...when... } \omega = 0 \tag{5}$$

The limit of the first equation in Equation (4) above as ω goes to zero is...

$$\begin{aligned}\lim_{\omega\to 0} f'(\omega) &= \alpha\,(1-\tau)\,\eta\,N_0 \left[n\,\mathrm{Exp}\left\{0\times n\right\} - m\,\mathrm{Exp}\left\{0\times m\right\}\right]\\ &= \alpha\,(1-\tau)\,\eta\,N_0 \left[n\times 1 - m\times 1\right[\\ &= \alpha\,(1-\tau)\,\eta\,N_0 \left[n - m\right]\end{aligned} \tag{6}$$

The limit of the second equation in Equation (4) above as ω goes to zero is...

$$\lim_{\omega \to 0} g'(\omega) = 1 \tag{7}$$

Using Equations (5), (6) and (7) above, the solution to Equation (1) above via L'Hopital's Rule is...

$$I_{m,n} = \lim_{\omega \to 0} \frac{f'(\omega)}{g'(\omega)} = \alpha\,(1-\tau)\,\eta\,N_0 \Bigg[n-m\Bigg] \Bigg/ 1 = \alpha\,(1-\tau)\,\eta\,N_0 \Bigg[n-m\Bigg] \tag{8}$$

References

[1] Gary Schurman, *Non-Interest-Bearing Liabilities - Growth Rate is Constan,* May, 2025